Eur. Phys. J. Plus (2019) **134**: 266
DOI 10.1140/epjp/i2019-12621-3

The European Physical Journal Plus

Regular Article

Higher-order approximate periodic solution for the oscillator with strong nonlinearity of polynomial type

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Received: 29 November 2018
Published online: 17 June 2019

Abstract. In this paper the harmonic balance method (HBM) is adopted for solving a special group of oscillators with strong nonlinear damping and elastic forces. The nonlinearity is of polynomial type. The motion is described with a strong nonlinear differential equation, whose approximate solution is assumed as a suitable sum of trigonometric functions. To find the most convenient combination of trigonometric functions as the probe function is the most important part of this investigation. Introducing the procedure of equating the terms with the same order of the trigonometric functions to zero, the problem is transformed into solving a system of nonlinear algebraic equations. Solving these equations the parameters of the solution up to high-order approximation are obtained. In the paper, the suggested solving procedure is applied for two nonlinear oscillator problems: free vibrations of a restrained uniform beam carrying an intermediate lumped mass and of a particle on a rotating parabola. The obtained approximate analytic solutions are compared with the already published results and with the numerically obtained solution. The solution up to third-order approximation is calculated. It is proved that the HBM with the suggested function gives more accurate results than the previous applied ones. Besides, the difference between the third-order approximate analytical solution and the numerical one is negligible. The method works well for different, even high, values of initial amplitudes.

1 Introduction

In the last three decades, a great attention has been devoted to the study of strong nonlinear oscillators and their copious application in the area of life science, physical science and engineering. These oscillators describe the motion of many physical phenomena and are modeled using strong nonlinear differential equations [1]. To solve these equations is not an easy task. For this aim, a number of novel mathematical methods have been introduced which are fairly more straightforward in analysis of such systems [2–4]. Let us mention some of them: generalized hyperbolic perturbation method [5], analytical approximate solutions [6], max-min approach [7], spreading residue harmonic balance method [8, 9], global residue harmonic balance method [10–13], Newton-harmonic balancing approach [14,15], energy balance method [16,17], frequency amplitude formulation [18,19], variational iteration method [20], variational approach [21–23], Hamiltonian approach [24,25], homotopy analysis approach [26] and so on.

The harmonic balance method (HBM), as one of the most general methods, provides approximate solutions for nonlinear problems [27–30]. It is convenient and effective for obtaining analytical approximate solutions not only to weak but also strongly nonlinear ordinary differential equations (ODEs) describing oscillatory systems. Namely, the method has several advantages, including, in particular, that it is applicable to ODEs of any order and does not require the nonlinearity to be small. However, very often there is a problem with accuracy of the obtained approximate solution. The accuracy depends on the efficiency of the investigator to choose the proper form of the solution.

The main object of the present work is to employ higher-order harmonic balance method (HBM) as a novel analytical method to obtain the approximate frequency and the corresponding periodic solutions for a special type of

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strong nonlinear oscillator with strong nonlinear elastic and damping forces. The nonlinearity in the oscillator is of polynomial type. We investigate the oscillator described with the differential equation

$$\ddot{x}\left(1+\sum_{n=1}^{\infty} b_{2n-1}x^{2n}\right)+\sum_{m=1}^{\infty} a_{2m-1}x^{2m-1}+\sum_{n=1}^{\infty} nb_{2n-1}x^{2n-1}\dot{x}^2=0, \tag{1}$$

whose conservation law is

$$\frac{\dot{x}^2}{2}\left(1+\sum_{n=1}^{\infty} b_{2n-1}x^{2n}\right)+\sum_{m=1}^{\infty} a_{2m-1}\frac{x^{2m}}{2m}=K=\text{const.}, \tag{2}$$

where a_{2m-1} and b_{2n-1} are constant parameters. The constant K has to satisfy the initial conditions of the problem

$$x(0)=A, \qquad \dot{x}(0)=0, \tag{3}$$

where A is the initial displacement. Equation (1) is the generalized version of the differential equations which describe the free vibration of a restrained uniform beam carryng an intermidiate lumped mass, and also the motion of the particle on a rotating parabola (see Nayfeh and Mook [31]). It is worth to say that, in [31], only the weak nonlinear systems are considered and the methods for approximate solution of the weak nonlinear differential equation are applied.

In this paper the extension of the model is done by introducing the strong nonlinearity. The aim of the paper is to obtain high-order accurate approximate analytical solution of eq. (1). For solving eq. (1) the method of harmonic balance is applied. Instead of the already assumed solution in the form of a trigonometric function, in this paper the special combination of the trigonometric functions is introduced which is suitable for determination of the high-order approximate solution. The obtained solution is compared with numerical one. The solving procedure, suggested in this paper, is tested on two already published problems.

The paper is divided into 5 sections. After the introduction, in sect. 2, the harmonic balance method is adopted for solving the strong nonlinear differential equation (1) based on the newly introduced trial solution which is the combination of the trigonometric functions. The procedure for frequency and parameter calculation for the higher-order approximate solution is developed. In sect. 3, the method is applied for solving of the oscillatory motion of the particle on a rotating parabola while, in sect. 4, for the free vibration of a restrained uniform beam carrying an intermediate lumped mass. The solutions up to the third-order approximation are calculated. In sect. 5, the obtained results are discussed. The HBM approximate solutions are compared with previously published analytical ones. Besides, the obtained solutions are compared with numerically calculated ones. Advantages and disadvantages of the suggested analytical method are discussed. The paper ends with conclusions.

2 Adopted HBM for higher-order approximate solution

HBM [8,32] is a powerful analytical solving method for the strong nonlinear oscillators described with the strong nonlinear differential equation. The trial solution of the problem is assumed in the form of Fourier series with a finite number of harmonic terms. The assumed solution is substituted into the given equation and the separation of the terms with the same order of the trigonometric function is done. Equating the so-obtained terms with zero, a set of algebraic equations is obtained. Solving the equations, the unknown coefficients of the solution are obtained. Due to complexity of the problem, usually the first-order HBM [33,34] is applied where only the first term of series expansion is taken into consideration. Unfortunately, the obtained solution is not accurate enough.

Nevertheless, in spite of the fact that the HBM seems quite simple, the main difficulty is how to assume the approximate solution of the problem.

For eq. (1), we assume the i-th–order approximate solution in the form

$$x_i=A\left(\cos(\omega t)-\sum_{i=1}^{\infty} u_i(\cos(\omega t)-\cos(2i-1)\omega t)\right), \tag{4}$$

where ω and coefficients u_i are unknown values. The number of unknowns depend on the order of approximation. Thus, for $i=1$ according to (4) the first-order solution is

$$x=A\cos(\omega t), \tag{5}$$

where $\omega=\omega_1$ is the only unknown value.

If $i=2$, the second-order approximate solution follows:

$$x=A\left(\cos(\omega t)-u_2\cos(\omega t)+u_2\cos(3\omega t)\right), \tag{6}$$

where two unknown values, $\omega = \omega_2$ and u_2, exist. Namely, substituting (6) into (1) and equating the terms with $\cos(\omega t)$ and $\cos(3\omega t)$, respectively, to zero, a system of two algebraic equations follows. Solving these equations, ω_2 and u_2 are obtained.

Similar, for $i = 3$, *i.e.*, the third-order approximate solution yields

$$x = A\left(\cos(\omega t) - u_2\cos(\omega t) + u_2\cos(3\omega t)\right) - u_3\cos(\omega t) + u_3\cos(5\omega t), \tag{7}$$

with three unknown values u_2, u_3 and $\omega = \omega_3$. Thus, the i-th–order approximate solution has i unknown values: ω and $i-1$ unknown constants u_i.

To determine the unknown values it is necessary to introduce the assumed solution into eq. (1) and to separate the terms multiplied with the same trigonometric function $\cos(\omega t)$, $\cos(3\omega t)$, ..., up to $\cos((2i-1)\omega t)$, where i corresponds to the order of approximation. Equating the so-obtained relations with zero, we obtain i algebraic equations. Solving this system of algebraic equations the unknown values are obtained.

2.1 General solution in the first approximation

Substituting the suggested first-order solution (5) into (1) gives

$$\omega_1^2\left(\cos(\omega_1 t) + \sum_{n=1}^{\infty} b_{2n-1}A^{2n}\cos^{2n+1}(\omega_1 t)\right) = \sum_{m=1}^{\infty} a_{2m-1}A^{2m-2}\cos^{2m-1}(\omega_1 t)$$
$$+\sum_{n=1}^{\infty} nb_{2n-1}A^{2n}\cos^{2n-1}(\omega_1 t) - \sum_{n=1}^{\infty} nb_{2n-1}A^{2n}\cos^{2n+1}(\omega_1 t).$$

Using the series expansion of the trigonometric functions and omitting the terms higher than the first order, the following relation is obtained:

$$\cos(\omega_1 t): \quad \omega_1^2\left(1 + \sum_{n=1}^{\infty} b_{2n-1}p_n A^{2n}\right) = \sum_{m=1}^{\infty} a_{2m-1}q_m A^{2m-2}$$
$$+\sum_{n=1}^{\infty} nb_{2n-1}q_n A^{2n} - \sum_{n=1}^{\infty} nb_{2n-1}p_n A^{2n}, \tag{8}$$

where q_m, q_n and p_n are coefficients of the Fourier series expansion of the trigonometric functions. Hence, the unknown frequency of vibration in the frirst approximation is

$$\omega_1 = \sqrt{\frac{\sum_{m=1}^{\infty} a_{2m-1}q_m A^{2m-2} + \sum_{n=1}^{\infty} n\left(b_{2n-1}q_n - b_{2n-1}p_n\right)^{2n}}{1 + \sum_{n=1}^{\infty} b_{2n-1}p_n A^{2n}}}. \tag{9}$$

Using the suggested procedures the constants in the second- and higher-order approximate solutions are determined.

2.2 General solution in the i-th approximation

Inserting the suggested i-th–order solution (4) into (1) leads to

$$\left(-A\omega^2\cos(\omega t) - \sum_{i=1}^{\infty} u_i\left(\omega^2\cos(\omega t) - (2i-1)^2\omega^2\cos(2i-1)(\omega t)\right)\right)$$
$$\left(1 + \sum_{n=1}^{\infty} b_{2n-1}\left(A\left(\cos(\omega t) - \sum_{i=1}^{\infty} u_i\left(\cos(\omega t) - \cos(2i-1)(\omega t)\right)\right)\right)^{2n}\right)$$
$$+\sum_{m=1}^{\infty} a_{2m-1}\left(A\left(\cos(\omega t) - \sum_{i=1}^{\infty} u_i\left(\cos(\omega t) - \cos(2i-1)(\omega t)\right)\right)\right)^{2m-1}$$
$$+\sum_{n=1}^{\infty} nb_{2n-1}\left(A\left(\cos(\omega t) - \sum_{i=1}^{\infty} u_i\left(\cos(\omega t) - \cos(2i-1)(\omega t)\right)\right)\right)^{2n-1}$$
$$\omega^2\left(A\left(-\sin(\omega t) + \sum_{i=1}^{\infty} u_i\left(\sin(\omega t) - (2i-1)\sin(2i-1)\omega t\right)\right)\right)^2 = 0. \tag{10}$$

Separating the terms with the same trigonometric functions and equating them with zero, we obtain a system of i algebraic equations

$$\begin{aligned}
&\cos(\omega t): \quad f_1\left(\omega^2, u_2, u_3 \ldots, u_i\right) = 0;\\
&\cos(3\omega t): \quad f_2\left(\omega^2, u_2, u_3 \ldots, u_i\right) = 0;\\
&\ldots\\
&\cos((2i-1)\omega t): \quad f_i\left(\omega^2, u_2, u_3 \ldots, u_i\right) = 0.
\end{aligned}$$

Solving the set of i coupled equations the values of $\omega^2, u_2, u_3 \ldots, u_i$ are obtained. Substituting these values into (4), the i-th–order approximate solution is obtained.

Two practical examples are presented to illustrate the solution procedure and to show the effectiveness of the proposed method.

3 Nonlinear dynamics of a particle on a rotating parabola

As the first example, the motion of a particle on a rotating parabola is considered. The governing equation of motion [31] and initial conditions are

$$\left(1+4q^2x^2\right)\ddot{x} + 4q^2x\dot{x}^2 + \Delta x = 0, \qquad x(0) = A, \qquad \dot{x}(0) = 0, \tag{11}$$

where q and Δ are constant values. Insert a new independent variable, $\tau = \omega t$, eq. (11) shifts to

$$\left(1+4q^2x^2\right)\omega^2 x'' + 4q^2\omega^2 x x'^2 + \Delta x = 0, \qquad x(0) = A, \qquad x'(0) = 0. \tag{12}$$

According to the previously mentioned procedure, the solution in the first, second and third approximation is calculated.

3.1 First-order harmonic balance method

Let us assume the first-order approximate solution of eq. (12) in the form (5). Substituting (5) into (12) and equating the coefficient of $\cos(\omega_1 t)$ with zero, we obtained the algebraic equation

$$\Delta - \omega_1^2 - 2A^2q^2\omega_1^2 = 0. \tag{13}$$

Solving eq. (13) the first-order analytical approximation of ω_1 is obtained as follows:

$$\omega_1 = \sqrt{\frac{\Delta}{1+2A^2q^2}}\,, \tag{14}$$

which is identical to the result in [35–37]. Hence, the approximate analytical periodic solution of eq. (11) can then be expressed as

$$x = A\cos\left(\sqrt{\frac{\Delta}{1+2A^2q^2}}\,t\right). \tag{15}$$

3.2 Second-order harmonic balance method

As was previously suggested, the second-order approximate solution is chosen in the form (6). Substituting (6) into eq. (12) and equating the coefficients of $\cos(\omega_2 t)$ and also of $\cos(3\omega_2 t)$, to zero, we get the following two equations:

$$\Delta - u_2\Delta - \omega_2^2 - 2A^2q^2\omega_2^2 + u_2\omega_2^2 - 14A^2q^2u_2^2\omega_2^2 + 16A^2q^2u_2^3\omega_2^2 = 0, \tag{16}$$

$$u_2\Delta - 2A^2q^2\omega_2^2 - 9u_2\omega_2^2 - 14A^2q^2u_2\omega_2^2 + 34A^2q^2u_2^2\omega_2^2 - 36A^2q^2u_2^3\omega_2^2 = 0. \tag{17}$$

Solving the above equations the unknown constants u_2 and ω_2 are determined as

$$u_2 = \frac{3}{5} - \frac{120A^2q^2 - 144A^4q^4}{30\sqrt[3]{630}A^2q^2(\Delta_1)^{\frac{1}{3}}} + \frac{(\Delta_1)^{\frac{1}{3}}}{5\sqrt[3]{6}A^2q^2}\,, \tag{18}$$

$$\omega_2 = \sqrt{\frac{(1-u_2)\Delta}{1+2A^2q^2 - u_2 + 14A^2q^2u_2^2 - 16A^2q^2u_2^3}}\,, \tag{19}$$

where

$$\Delta_1 = -450A^4q^4 - 63A^6q^6 + 5\sqrt{3}\sqrt{640A^6q^6 + 15844A^8q^8 + 3672A^{10}q^{10} - 1053A^{12}q^{12}}.$$

The terms with u_2 make correction to the frequency obtained by applying of the first-order HBM. The approximate analytical periodic solution can then be expressed as

$$x = A\left(\cos(\omega_2 t) - u_2\cos(\omega_2 t) + u_2\cos(3\omega_2 t)\right). \tag{20}$$

3.3 Third-order harmonic balance method

To find the third-order analytical approximation, we insert the solution (7) into (11) and expand the resulting expression in a trigonometric series. Setting the coefficients of $\cos(\omega_2 t)$, $\cos(3\omega_2 t)$, and $\cos(5\omega_2 t)$ to zero one can obtain three linear algebraic equations. Solving these equations, three unknown constants u_2, u_3 and ω_3 are determined. It should be noted that the symbolic mathematical, Mathematica software commands are used to obtain the above coefficients. The relations u_2, u_3 and ω_3 are not presented here as they require too much space. Nevertheless, in the following numerical example the values of u_2, u_3 and ω_3 will be calculated according to their analytical expression and reported in the text.

4 Vibration of a restrained uniform beam carrying an intermediate lumped mass

Let us consider the free vibration of a restrained uniform beam carrying an intermediate lumped mass which was first introduced by Hamdan and Shabaneh [38]:

$$\ddot{x} + \lambda x + \varepsilon_1 x^2\ddot{x} + \varepsilon_1 x\dot{x}^2 + \varepsilon_2 x^4\ddot{x} + 2\varepsilon_2 x^3\dot{x}^2 + \varepsilon_3 x^3 + \varepsilon_4 x^5 = 0, \tag{21}$$

with the initial conditions

$$x(0) = A, \qquad \dot{x}(0) = 0.$$

Here λ and $\varepsilon_1 - \varepsilon_4$ are constant values. By introducing a new variable $\tau = \omega t$ and substituting it into eq. (21), one obtains

$$\omega^2 x'' + \lambda x + \varepsilon_1\omega^2 x^2 x'' + \varepsilon_1\omega^2 x x'^2 + \varepsilon_2\omega^2 x^4 x'' + 2\varepsilon_2\omega^2 x^3 x'^2 + \varepsilon_3 x^3 + \varepsilon_4 x^5 = 0, \tag{22}$$

where

$$x(0) = A, \qquad x'(0) = 0.$$

4.1 First-order harmonic balance method

For the first-order analytical approximation solution (5), we obtain the algebraic equation with unknown value ω_1,

$$16\lambda - 16\omega_1^2 - 8A^2\omega_1^2\varepsilon_1 - 6A^4\omega_1^2\varepsilon_2 + 12A^2\varepsilon_3 + 10A^4\varepsilon_4 = 0. \tag{23}$$

Solving eq. (23), the first-order approximate angular frequency becomes

$$\omega_1 = \sqrt{\frac{8\lambda + 6A^2\varepsilon_3 + 5A^4\varepsilon_4}{8 + 4A^2\varepsilon_1 + 3A^4\varepsilon_2}}, \tag{24}$$

and the approximate analytical periodic solution of eq. (22) is expressed as

$$x = A\cos\left(\sqrt{\frac{8\lambda + 6A^2\varepsilon_3 + 5A^4\varepsilon_4}{8 + 4A^2\varepsilon_1 + 3A^4\varepsilon_2}}t\right). \tag{25}$$

4.2 Second-order harmonic balance method

Introducing the second-order approximate solution (6) into (22) and equating the separated terms of $\cos(\omega_2 t)$ and $\cos(3\omega_2 t)$ to zero, two equations follow:

$$\begin{aligned}
&\lambda - u_2\lambda - \omega_2^2 + u_2\omega_2^2 - \frac{1}{2}A^2\omega_2^2\varepsilon_1 - \frac{7}{2}A^2u_2^2\omega_2^2\varepsilon_1 - \frac{3}{8}A^4\omega_2^2\varepsilon_2 - \frac{5}{16}A^4u_2\omega_2^2\varepsilon_2 - \frac{13}{4}A^4u_2^2\omega_2^2\varepsilon_2\\
&+\frac{3}{4}A^2\varepsilon_3 - \frac{3}{2}A^2u_2\varepsilon_3 + \frac{9}{4}A^2u_2^2\varepsilon_3 + \frac{5}{8}A^4\varepsilon_4 - \frac{25}{16}A^4u_2\varepsilon_4 + \frac{15}{4}A^4u_2^2\varepsilon_4 = 0,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\lambda u_2 - 9u_2\omega_2^2 - \frac{1}{2}A^2\omega_2^2\varepsilon_1 - \frac{7}{2}A^2u_2\omega_2^2\varepsilon_1 + \frac{17}{2}A^2u_2^2\omega_2^2\varepsilon_1 - \frac{7}{16}A^4\omega_2^2\varepsilon_2 - \frac{31}{16}A^4u_2\omega_2^2\varepsilon_2\\
&+\frac{13}{2}A^4u_2^2\omega_2^2\varepsilon_2 + \frac{1}{4}A^2\varepsilon_3 + \frac{3}{4}A^2u_2\varepsilon_3 + \frac{5}{16}A^4\varepsilon_4 + \frac{5}{16}A^4u_2\varepsilon_4 - \frac{5}{2}A^4u_2^2\varepsilon_4 = 0.
\end{aligned} \tag{27}$$

Solving eqs. (26) and (27) simultaneously, we obtain

$$u_2 = \frac{\Delta_2 - \sqrt{\Delta_2^2 - 4\Delta_3\Delta_4}}{2\Delta_2} \tag{28}$$

and

$$\omega_2 = \sqrt{\frac{16u_2\lambda + 4A^2\varepsilon_3 + 12A^2u_2\varepsilon_3 - 36A^2u_2^2\varepsilon_3 + 32A^2u_2^3\varepsilon_3 + 5A^4\varepsilon_4 - 40A^4u_2^2\varepsilon_4 + 100}{144u_2 + 8A^2\varepsilon_1 + 56A^2u_2\varepsilon_1 - 136A^2u_2^2\varepsilon_1 + 144A^2u_2^3\varepsilon_1}}, \tag{29}$$

where

$$\begin{aligned}
\Delta_2 &= 512\lambda + 160A^2\lambda\varepsilon_1 + 72A^4\lambda\varepsilon_2 + 400A^2\varepsilon_3 + 96A^4\varepsilon_1\varepsilon_3 + 28A^6\varepsilon_2\varepsilon_3 + 360A^4\varepsilon_4\\
&\quad + 80A^6\varepsilon_1\varepsilon_4 + 20A^8\varepsilon_2\varepsilon_4,\\
\Delta_3 &= -32A^2\lambda\varepsilon_1 - 28A^4\lambda\varepsilon_2 + 16A^2\varepsilon_3 - 15A^6\varepsilon_2\varepsilon_3 + 20A^4\varepsilon_4 - 10A^6\varepsilon_1\varepsilon_4 - 10A^8\varepsilon_2\varepsilon_4,\\
\Delta_4 &= 512\lambda + 768A^2\lambda\varepsilon_1 + 560A^4\lambda\varepsilon_2 + 672A^2\varepsilon_3 + 656A^4\varepsilon_1\varepsilon_3 + 440A^6\varepsilon_2\varepsilon_3 + 720A^4\varepsilon_4\\
&\quad + 560A^6\varepsilon_1\varepsilon_4 + 360A^8\varepsilon_2\varepsilon_4.
\end{aligned}$$

Thus the second-order approximation solution of eq. (21) is

$$x = A\left(\cos(\omega_2 t) - u_2\cos(\omega_2 t) + u_2\cos(3\omega_2 t)\right), \tag{30}$$

where u_2 and ω_2 are, respectively, given by eqs. (28) and (29).

4.3 Third-order harmonic balance method

The explanation for obtaining the third-order analytical approximation is the same as that described in sect. 3.3. Expressions for u_2, u_3 and ω_3 are not presented due to their complexity. However, in the following text they will be calculated for certain numerical values.

5 Results and discussion

It is of interest to compare the solutions given in sects. 3 and 4 with analytical ones, obtained applying another solving procedures, and with numerical solution of eq. (11) and eq. (21) for initial conditions (12) and (22), respectively.

In table 1, the approximate and numerically obtained frequencies corresponding to various parameters in eq. (11) are shown. It is concluded that amplitude-frequency formulation (AFF) [35] and variational approach (VA) [36,37] give the solution which corresponds to the first-order HBM solution (15) obtained for the example 1 in sect. 3. Table 1 shows that as the order of the approximations increases the difference between that solution and the first-order solution increases. At the same time, the difference this solution is closer to the numerically obtained result. Namely, for the third-order approximation the frequency error is almost zero. In fig. 1, the third-order approximate solution is compared with numerical solution of (11). It can be seen that the difference is negligible even for high values of nonlinearity and initial amplitude.

Table 1. Comparison of the approximate and numerically obtained frequencies corresponding to various parameters in eq. (11).

A	q	Δ	$\omega_{AFF}^{[35]} = \omega_{VA}^{[36,37]}$ (Error %)	ω_1 (Error %)	ω_2 (Error %)	ω_3 (Error %)	ω_{Num}
0.5	1	0.5	0.5774	0.5774	0.5801	0.5815	0.5815
			(0.7135)	(0.7135)	(0.24076)	(0.0000)	
0.5	0.5	2	1.3333	1.3333	1.3343	1.3344	1.3344
			(0.0774)	(0.0774)	0.00749	(0.0000)	
1	0.8	1.5	0.8111	0.8111	0.8181	0.8284	0.8288
			(2.1399)	(2.1399)	(1.29102)	(0.04826)	
1	0.7	0.5	0.5025	0.5025	0.5064	0.5108	0.5108
			(1.6300)	(1.6300)	(0.86139)	(0.0000)	
1.5	0.5	2	0.9701	0.9701	0.9782	0.9885	0.9888
			(1.8836)	(1.8836)	(1.07201)	(0.03034)	
1.5	0.3	2.5	1.3339	1.3339	1.3390	1.3411	1.3410
			(0.5298)	(0.5298)	(0.14914)	(0.00746)	
2	0.2	4	1.7408	1.7408	1.7458	1.7473	1.7473
			(0.3725)	(0.3725)	(0.08585)	(0.0000)	
2	0.4	1	0.6623	0.6623	0.6680	0.6764	0.6767
			(2.1399)	(2.1399)	(1.28565)	(0.04433)	

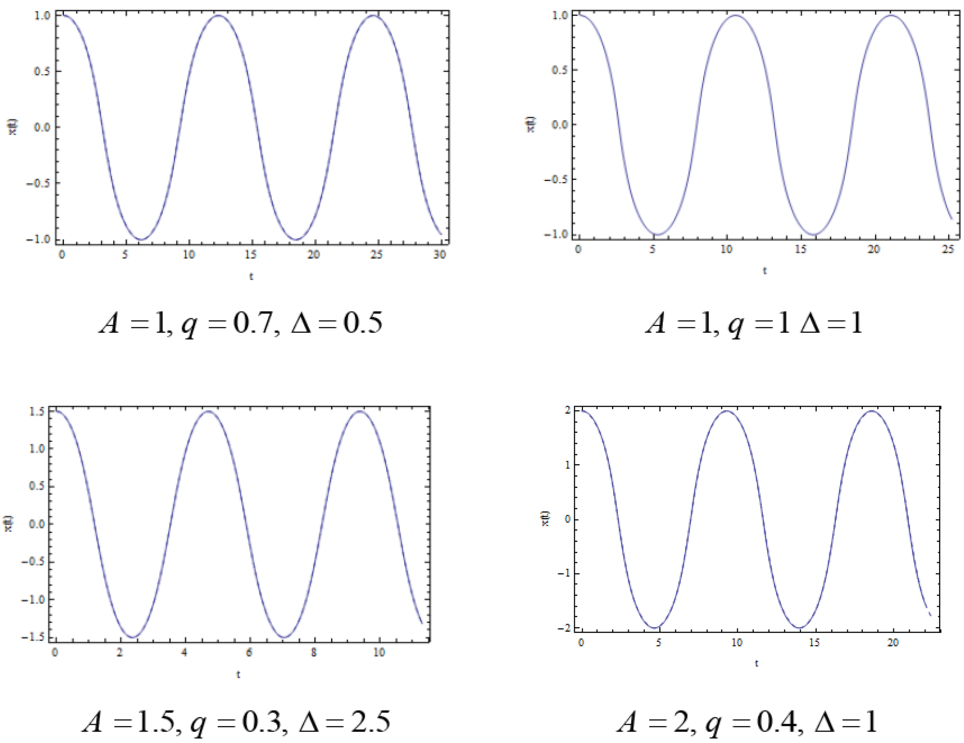

Fig. 1. Comparison of the approximate (- - -) and numerical solutions (—).

Table 2. Comparison of the approximate and exact frequencies corresponding to various parameters in eq. (21) for $\lambda = 1$.

Mode [42]	A	ω_{EBM} [39] (Error %)	ω_{VA} [38] $= \omega_{HPM}$ [40] (Error %)	ω_{HAM} [41] (Error %)	ω_1 (Error %)	ω_2 (Error %)	ω_3 (Error %)	ω_{Exact}
1	1	1.0123	1.00705	1.01232	1.00706	1.01004	1.01004	1.01015
		(0.21284)	(0.30664)	(0.21482)	(0.30589)	(0.01088)	(0.01088)	
2	0.5	0.9350	0.93255	0.938636	0.932556	0.934943	0.935108	0.93639
		(0.14844)	(0.40947)	(0.23986)	(0.40944)	(0.15453)	(0.13691)	
3	0.2	0.965613	0.965469	0.966516	0.965469	0.965827	0.965845	0.96664
		(0.10624)	(0.12144)	(0.01283)	(0.12114)	(0.08411)	(0.08224)	
4	0.3	0.860678	0.859702	0.871382	0.859702	0.863147	0.864155	0.86426
		(0.41446)	(0.52752)	(0.82406)	(0.52739)	(0.12878)	(0.01215)	

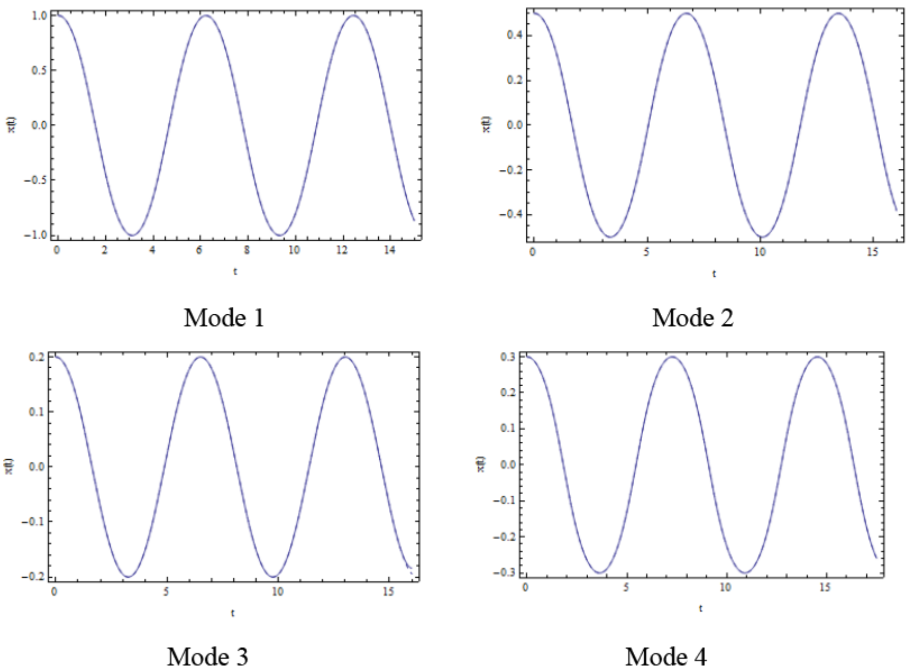

Fig. 2. Comparison of the approximate (- - -) and numerical solutions (—).

In table 2, for the example 2 explained in sect. 4, the frequency of vibration calculated in the first (23), second (28) and third approximation is compared with the VA [38], the energy balance method (EBM) [39], homotopy perturbation method (HPM) [40] and HAM [41] solutions and also with numerically obtained result for eq. (21). The results are compared and the error according to the numerical solution is calculated. The significant difference between the third-order approximate HBM solutions to the previously obtained analytical results is evident. Otherwise, the HBM solution in the third approximation tends to the numerically obtained result. The numerical solution is calculated applying the fourth-order Runge-Kutta procedure. In fig. 2, the analytically obtained third-order approximate solution and the numerically obtained displacement-time diagrams are plotted. The difference between them is negligible.

6 Conclusion

In this paper, a high-order approximation harmonic balance method based on a new type of trial solution is developed. The method is suitable for solving of a special kind of the oscillator with strong nonlinearity of polynomial type. The third-order approximation is applied to obtain approximate periodic solution for the free vibrations of a restrained uniform beam carrying an intermidiate lumped mass and for a particle on a rotating parabola. The following conclusions are drawn.

1) The suggested procedure with the introduced trial function (4) is suitable to give the accurate solution for the strong nonlinear oscillator (1).
2) Results obtained with higher-order harmonic balance method quickly converge to the exact solution. Difference between analytically obtained high-order HBM solution and the numerically calculated one is negligible.
3) The frequency of vibration calculated up to the third order of approximation is almost equal to the numerically calculated value even for high nonlinearities and wide range of initial conditions.
4) The advantage of the method is its simplicity in comparison to the other methods. Using this certain type of trial solution, the differential equation of motion is transformed into a system of algebraic equations which are easy to be solved although the calculated coefficients are very complex.
5) The disadvantage of the introduced trial solution is that it is suitable only for certain oscillators with strong nonlinearity of polynomial type.

Conflict of interest

The authors declare that they have no conflict of interest.

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